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# Multiscale expansion on the lattice and integrability of partial difference equations

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## Abstract

We conjecture an integrability and linearizability test for dispersive  $\mathbb{Z}^2$ -lattice equations by using a discrete multiscale analysis. The lowest order secularity conditions from the multiscale expansion give a partial differential equation of the form of a nonlinear Schrödinger (NLS) equation. If the starting lattice equation is integrable then the resulting NLS-type equation turns out to be integrable, while if the starting equation is linearizable we get a linear Schrödinger equation. On the other hand, if we start with a non-integrable lattice equation the resulting equation can be both integrable and non-integrable. This conjecture is confirmed by many examples.

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## 1. Introduction

By a difference equation we mean a functional relation, linear or nonlinear, between functions calculated at different points of a lattice [3, 4, 9, 21, 35]. These systems appear in many applications. First of all they can be written as discretizations of a differential equation when one is trying to solve it with the help of computer. In such a case one reduces the differential equation to a recurrence relation. On the other hand we can consider dynamical systems defined on a lattice, i.e. systems where the real independent fields depend on a set of independent variables which run through the integers and through the reals. For example we

can consider the differential–difference equation:

$$\frac{d^2 u_n(t)}{dt^2} = F(t, u_n(t), u_{n-1}(t), \dots, u_{n-a}(t), u_{n+1}(t), \dots, u_{n+b}(t)),$$

with  $n, a, b \in \mathbb{N}$ ,  $t \in \mathbb{R}$ . This kind of equations can appear in many different settings, including the evolution of many body problems, the study of crystals, biological and economical systems and so on.

The world of differential–difference equations is indeed much richer than that of partial differential equations as one can see in [28], where it is shown that almost any Hamiltonian network of weakly coupled oscillators has a ‘breather’ solution, while the existence of breathers for a nonlinear wave equation is rare. These results imply that the discrete world can be richer in interesting solutions and thus worthwhile studying by itself.

In the discrete world one can easily use computers to solve the relevant discrete maps but their solution is always approximate due to numerical rounding errors. So the possibility of treating discrete models by exact techniques is very important as in this way new phenomena may be uncovered. Exact techniques are naturally associated with integrable models; therefore to find techniques for showing the integrability of discrete systems is a challenging task.

Various different criteria have been proposed as tests for integrability in the discrete setting. One of the earliest proposals for *ordinary* difference equations was the singularity confinement test of Grammaticos *et al* [10], which has proved to be an extremely useful tool for isolating discrete Painlevé equations. Hietarinta and Viallet, after discovering some non-integrable equations with the singularity confinement property, were led to introduce the algebraic entropy condition for integrability of rational maps [15] (the phenomenon of weak degree growth of integrable maps had been studied earlier by Veselov [36, 37] and it is connected with the Arnold complexity [1]). Ablowitz *et al* extended the Painlevé property to difference equations using Nevanlinna theory [2] by considering the asymptotic growth of meromorphic solutions at infinity. Roberts and Vivaldi [32] have studied the distribution of orbit lengths in rational maps reduced to finite fields  $\mathbb{F}_p$  for different primes  $p$ , in order to identify integrable cases of such maps. The algebraic entropy method has been applied to detect integrability also in the case of two-dimensional *partial* difference equations (or  $\mathbb{Z}^2$ -lattice equations) [16, 38].

In the continuous setting multiscale techniques [33, 34] have proved to be important tools for finding approximate solutions to many physical problems by reducing a given dispersive nonlinear partial differential equation to a simpler equation, which is often integrable [6]. These multiscale expansions are structurally strong and can be applied to both integrable and non-integrable systems. Zakharov and Kuznetsov in the introduction of their article [40] say: ‘*If the initial system is not integrable, the result can be both integrable and non-integrable. But if we treat the integrable system properly, we again must get from it an integrable system*’. Calogero and Eckhaus [6] used similar ideas starting from a class of hyperbolic systems to prove in 1987 the necessary conditions for the integrability of dispersive nonlinear partial differential equations. Later, Degasperis and Procesi [8] introduced the notion of *asymptotic integrability of order  $n$*  by requiring that the multiscale expansion be verified up to a fixed order  $n$ .

Recently a few attempts to carry over this approach to difference–difference and differential–difference equations have been proposed [5, 11, 12, 18, 22–24, 26]. In [18, 23, 24, 26] we developed a multiscale expansion technique on the lattice which, starting from dispersive integrable  $\mathbb{Z}^2$ -lattice equations, provided non-integrable  $\mathbb{Z}^2$ -lattice equations, showing that the integrable system was not treated properly.

Later on, in [11], this problem has been solved by extending the previous results to *functions of infinite slowvaryness order* (see section 2 for details). This new technique easily fits for both difference–difference and differential–difference equations. The representative example considered in [11] was the lattice potential KdV equation. In such a case a proper representation of the discrete shift operators in terms of differential operators, which is indeed equivalent to a Taylor expansion, provides an integrable nonlinear Schrödinger (NLS) equation as the lowest order secularity conditions from the multiscale expansion. Results obtained performing the Taylor expansion of the shifted variables can be found in many papers, see for instance [7, 20].

The result found in [11] suggests the investigation of a discrete analogue of the Calogero–Eckhaus theorem [6], which claims that a necessary condition for the integrability of a dispersive nonlinear partial differential equation is that its lowest order of the multiscale expansion be an integrable differential equation. Here, a nonlinear partial differential (resp. difference) equation is said to be *integrable* if there exists a non-trivial Lax pair associated with the equation, while it is said to be *linearizable* if there exists a Miura transformation which maps it into a linear equation. For a rigorous definition of a Miura transformation on the lattice we refer to [39] (section 2.7).

In the present work, by using the formalism developed in [11], we propose the following conjecture:

**Conjecture.** *If a nonlinear dispersive  $\mathbb{Z}^2$ -lattice equation is integrable (resp. linearizable) then its lowest order multiscale reduction will be an integrable nonlinear (resp. linear) Schrödinger equation.*

To confirm the above conjecture we shall consider several examples, both integrable and non-integrable, linearizable and non-linearizable. Let us stress that the procedure developed in [11] and used here can be applied only to dispersive  $\mathbb{Z}^2$ -lattice equations (i.e., a non-trivial dispersion relation for the linear part must exist).

In section 2 we present a brief review of the discrete multiscale method. Then, in section 3 we apply the multiscale expansion to some difference–difference KdV-type and Toda-type equations. In section 4 we consider some linearizable equations, as the differential–difference Burgers equation and the Hietarinta equation, and also a non-linearizable difference–difference Burgers-type equation. Finally, section 5 is devoted to open problems and concluding remarks.

## 2. Basic formulae for the discrete multiscale analysis

The basic tool of the discrete reductive perturbation technique developed in [11] is a proper multiscale expansion which consists in considering various lattices and functions defined on them. The relation between the variation of a function on two different lattices of indices  $n$  and  $n_1$  is given by [19]

$$\Delta_n^j u_n = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} u_{n+i} = j! \sum_{i=j}^{\infty} \frac{P_{i,j}}{i!} \Delta_{n_1}^i u_{n_1}, \tag{1}$$

where  $u_n : \mathbb{Z} \rightarrow \mathbb{R}$  is a function defined on a lattice of index  $n \in \mathbb{Z}$  and  $u_{n_1} : \mathbb{Z} \rightarrow \mathbb{R}$  is the same function on a lattice of index  $n_1 \in \mathbb{Z}$ . According to equation (1), by the symbol  $\Delta_n$  we mean the standard forward difference of the function  $u_n$  with respect to its subscript, e.g.  $\Delta_n u_n = (T_n - 1)u_n = u_{n+1} - u_n$ , where  $T_n$  is the shift operator  $T_n u_n = u_{n+1}$ . The coefficients

$P_{i,j}$  in equation (1) are expressed in terms of the ratio of lattice spacing for the variable  $n_1$  with respect to that of variable  $n$ .

Equation (1) implies that a finite difference in the discrete variable  $n$  depends on an infinite number of differences on the variable  $n_1$ , e.g. the function  $u_{n+1}$  can be written as a combination of the functions  $u_i$ 's, for  $i$  varying on an infinite set of points of the lattice  $n_1$ . In [11] one has considered generalizations of formula (1) in order to deal with functions  $u_n = u_{n;\{n_i\}_{i=1}^K}$  depending on a finite number  $K$  of lattice variables  $n_i$  and with functions depending on two discrete indices, say  $n$  and  $m$ , thus dealing with  $\mathbb{Z}^2$ -lattice equations.

To get a reduction of a given difference equation onto a difference equation of order less than a fixed number, say  $\ell$ , one has to consider functions  $u_n$  of *slow varyness of order  $\ell$* , namely the space of those functions  $u_n$  such that  $\Delta_n^{\ell+1}u_n = 0$  or, equivalently,  $\Delta_{n_1}^{\ell+1}u_{n_1} = 0$  (see [26]). With this definition one can reduce the infinite series expansion (1) to a finite number of terms.

To deal with functions of infinite order of slow varyness one considers a formal expansion of the shift operator  $T_n$ . By introducing on the lattice of index  $n$  the real variable  $x = n\sigma_x$ , the shift operator  $T_x$  such that  $T_x u(x) = u(x + \sigma)$  can be formally written as

$$T_x = \exp(\sigma_x d_x) = \sum_{i=0}^{\infty} \frac{\sigma_x^i}{i!} d_x^i.$$

Introducing a formal derivative with respect to the index  $n$ , say  $\delta_n$ , we can define the discrete operator  $T_n$  as

$$T_n = \exp(\delta_n) = \sum_{i=0}^{\infty} \frac{\delta_n^i}{i!}. \tag{2}$$

The formal expansion (2) can be inverted, yielding

$$\delta_n = \ln T_n = \ln(1 + \Delta_n) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \Delta_n^i, \tag{3}$$

where  $\Delta_n$  is the discrete first right-difference operator with respect to the variable  $n$ , see equation (1). We refer to [11] for more technical details and decompositions with respect to different discrete derivatives.

To perform a multiscale expansion we need to consider functions defined on different lattices, thus depending on a fast lattice index  $n$  and on multiple slow-varying lattice indices  $n_i$ ,  $1 \leq i \leq K$ . The slow-varying lattice variables vary on a larger scale with respect to that of the original lattice of index  $n$ , and thus the transition from  $n$  to  $n_1$  corresponds to a coarse graining of the lattice. In the continuous limit, when the spacing between the lattice points goes to zero, this corresponds to the introduction of multiple continuous variables: given  $x \in \mathbb{R}$  we define the new variables  $x_i = \epsilon^i x$ ,  $0 < \epsilon \ll 1$ ,  $1 \leq i \leq K$ . By taking into account the above definitions we can introduce a function  $u_{n;\{n_i\}_{i=1}^K} = u(x; \{x_i\}_{i=1}^K)$  depending on a fast index  $n$  and  $K$  slow indices  $n_i = \epsilon^i n$ . Here  $\epsilon = 1/N$  and  $N$  is an integer number if we require that the slow indices have to be integer numbers.

At the continuous level, the total derivative  $d_x$  acting on functions  $u(x; \{x_i\}_{i=1}^K)$  is the sum of partial derivatives, i.e.  $d_x = \partial_x + \epsilon \partial_{x_1} + \epsilon^2 \partial_{x_2} + \mathcal{O}(\epsilon^3)$ . Consequently we can expand the total shift operator  $T_x$  in terms of the partial shift operators

$$T_x = \exp(\sigma_x d_x) = \exp(\sigma_x \partial_x) \exp(\epsilon \sigma_x \partial_x) \exp(\epsilon^2 \sigma_x \partial_{x_2}) \dots$$

At the discrete level, we can write

$$T_n = \exp(\delta_n) \exp(\epsilon \delta_{n_1}) \exp(\epsilon^2 \delta_{n_2}) \dots = T_n T_{n_1}^\epsilon T_{n_2}^{\epsilon^2} \dots,$$

with

$$\mathcal{T}_n = \sum_{i=0}^{\infty} \frac{\delta_n^i}{i!}, \quad \mathcal{T}_{n_1}^\epsilon = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \delta_{n_1}^i, \quad \mathcal{T}_{n_2}^{\epsilon^2} = \sum_{i=0}^{\infty} \frac{\epsilon^{2i}}{i!} \delta_{n_2}^i, \quad \dots,$$

where the  $\delta$ -operators are given in equation (3) and they act on a function  $u_{n; \{n_i\}_{i=1}^K}$  with respect to their subscript. From the previous formulae we deduce immediately that working with  $\delta$ -operators is equivalent, up to terms depending on  $\sigma_x$ , to performing a Taylor expansion of the discrete function  $u_{n; \{n_i\}_{i=1}^K}$ , e.g. for  $K = 2$  we have

$$\mathcal{T}_n u_{n; n_1, n_2} = \mathcal{T}_n \left[ 1 + \epsilon \delta_{n_1} + \frac{\epsilon^2}{2} \delta_{n_1}^2 + \epsilon^2 \delta_{n_2} + \mathcal{O}(\epsilon^3) \right] u_{n; n_1, n_2}.$$

In the following sections we shall discuss a list of examples of integrable, non-integrable, linearizable and non-linearizable dispersive lattice equations.

### 3. Multiscale expansion of discrete Toda-type and KdV-type equations

#### 3.1. An integrable difference–difference Toda equation

An integrable discrete-time Toda equation is given by [22]

$$\exp(u_{n,m} - u_{n,m+1}) - \exp(u_{n,m+1} - u_{n,m+2}) = a^2 [\exp(u_{n-1,m+2} - u_{n,m+1}) - \exp(u_{n,m+1} - u_{n+1,m})], \quad (4)$$

where  $a \in \mathbb{R}$  is a parameter related to discretization of the time variable. The above five-point  $\mathbb{Z}^2$ -lattice equation had been obtained by Hirota 30 years ago [17].

We can split equation (4) into a linear and nonlinear part by considering its small amplitude solutions, namely  $u_{n,m} = \epsilon w_{n,m}$ ,  $0 < \epsilon \ll 1$ ,  $\epsilon$  being a small parameter as that we considered in section 2. The linear part of equation (4) has a travelling wave solution of the form  $w_{n,m} = \exp\{i[\kappa n - \omega(\kappa)m]\}$ . Here the dispersion relation  $\omega = \omega(\kappa)$  obeys the following equation:

$$a^2(\Omega - \mathcal{K})^2 = \mathcal{K}(\Omega - 1)^2, \quad (5)$$

where we have introduced the quantities  $\mathcal{K} = \exp(i\kappa)$  and  $\Omega = \exp(-i\omega)$ ,  $\kappa, \omega \in \mathbb{R}$ .

As nonlinearity generates harmonics, we introduce the following expansion for the function  $w_{n,m}$ :

$$w_{n,m} = \sum_{\alpha \in \mathbb{Z}} \sum_{k=0}^{\infty} \epsilon^k w_k^{(\alpha)} e^{i\alpha(\kappa n - \omega m)}, \quad (6)$$

where  $w_k^{(-\alpha)} = \bar{w}_k^{(\alpha)}$ ,  $\bar{w}$  being the complex conjugate of  $w$ . Moreover, to avoid secularities we have to require that  $w_k^{(\alpha)} = w_k^{(\alpha)}(n_1, \{m_i\}_{i=1}^K)$ , where  $n_1$  and  $\{m_i\}_{i=1}^K$  are  $K + 1$  slow-varying lattice variables, namely  $n_1 = \epsilon n$  and  $m_i = \epsilon^i m$ ,  $1 \leq i \leq K$ .

The multiscale expansion of equation (4) provides several determining equations for the coefficients  $w_k^{(\alpha)}$ , obtained selecting the different powers of  $\epsilon$  and the different harmonics  $\alpha$ . Let us give here the main results obtained considering the lowest  $\epsilon$  and  $\alpha$  orders of this development.

At  $\mathcal{O}(1)$ , for  $\alpha = 0, 1$ , we find linear equations which are identically satisfied either directly or by taking into account the dispersion relation (5). For  $|\alpha| \geq 2$  one gets some linear equations whose unique solution is given by  $w_0^{(\alpha)} = 0$ .

At  $\mathcal{O}(\epsilon)$ , for the harmonics  $\alpha = 1, 2$ , we find the following equations:

$$(\tau_0 \delta_{n_1} + \delta_{m_1}) w_0^{(1)} = 0, \quad (7)$$

$$w_1^{(2)} = \tau_1 (w_0^{(1)})^2, \tag{8}$$

where

$$\tau_0 = -\frac{(\Omega - 1)(\Omega + \mathcal{K})}{2\Omega(\mathcal{K} - 1)}, \quad \tau_1 = \frac{\mathcal{K} + \infty}{2(\mathcal{K} - 1)}.$$

By solving equation (5) with respect to  $\omega = \omega(\kappa)$  one obtains that  $\tau_0 = v_g = d\omega/d\kappa$ , where  $v_g$  is the group velocity. The solution to equation (7) is given by  $w_0^{(1)}(n_1, \{m_i\}_{i=1}^K) = w_0^{(1)}(n_2, \{m_i\}_{i=2}^K)$  with  $n_2 = n_1 - v_g m_1$ . Equation (8) expresses  $w_1^{(2)}$  in terms of  $w_0^{(1)}$ . Moreover, as  $w_0^{(1)}$  is a function of  $n_2$  the same must hold for  $w_1^{(2)}$ , i.e.  $w_1^{(2)}(n_1, \{m_i\}_{i=1}^K) = w_1^{(2)}(n_2, \{m_i\}_{i=2}^K)$ .

We can now consider the  $\mathcal{O}(\epsilon^2)$ . For  $\alpha = 0$  we have

$$[(\tau_2 \delta_{m_1}^2 + \tau_3 (\delta_{n_1}^2 - 2\delta_{n_1} \delta_{m_1}))] w_0^{(0)} = \tau_4 (\delta_{n_1} - 2\delta_{m_1}) |w_0^{(1)}|^2, \tag{9}$$

with

$$\tau_2 = -\frac{6\Omega(\mathcal{K} - 1)(\Omega^2 - \mathcal{K})}{(\Omega - \mathcal{K})^2}, \quad \tau_3 = -\frac{6\mathcal{K}\Omega(\Omega - 1)^2}{(\Omega - \mathcal{K})^2}, \quad \tau_4 = 6(\Omega - 1)^2.$$

As  $w_0^{(1)}$  is a function of  $n_2$ , the same must be for  $w_0^{(0)}$ . Thus we can integrate equation (9) by requiring that its solution be bounded and we obtain

$$[(\tau_2 v_g^2 + \tau_3 (1 + 2v_g))] \delta_{n_2} w_0^{(0)} = \tau_4 (1 + 2v_g) |w_0^{(1)}|^2. \tag{10}$$

For  $\alpha = 1$  we find a secular equation for  $w_1^{(1)}$  which is solved by requiring that  $w_0^{(1)}$  satisfies the following equation:

$$(\tau_5 \delta_{m_2} + \tau_6 \delta_{n_2}^2) w_0^{(1)} = \tau_7 w_0^{(1)} \delta_{n_2} w_0^{(0)} + \tau_8 \bar{w}_0^{(1)} w_1^{(2)}, \tag{11}$$

where

$$\tau_5 = -\frac{12(\mathcal{K} - 1)(\Omega - 1)\Omega^2}{\Omega - \mathcal{K}}, \quad \tau_6 = \frac{3(\Omega^2 - \mathcal{K})(\Omega - 1)^2}{2(\mathcal{K} - 1)},$$

$$\tau_7 = 4\tau_6, \quad \tau_8 = -\frac{6(\mathcal{K} + 1)(\Omega^2 - \mathcal{K})(\Omega - 1)^2}{\mathcal{K}}.$$

Taking into account equations (8), (10) and (11) reduces to the following NLS-type equation for  $w_0^{(1)}$ :

$$i\delta_{m_2} w_0^{(1)} = \tau_9 \delta_{n_2}^2 w_0^{(1)} + \tau_{10} w_0^{(1)} |w_0^{(1)}|^2, \tag{12}$$

where

$$\tau_9 = \frac{\sin(\kappa + \omega) + \sin \omega - \sin(2\omega + \kappa)}{8(\cos \kappa - 1)},$$

$$\tau_{10} = \frac{\sin(2\omega + \kappa)(\cos \kappa + 5) - 10 \sin(\omega + \kappa/2) \cos(\kappa/2)}{4(\cos \kappa - 1)}.$$

Note that as  $\tau_9$  and  $\tau_{10}$  are real parameters, equation (12) is a real scaling of the NLS equation (when  $\ell = \infty$ ), i.e. it is integrable. If any of these coefficients has a non-zero imaginary part the equation would not be integrable, as it can be deduced, for example, from the integrability classifications in [29, p 214].

As noticed in [11], assuming a finite order of slow varyness in the NLS-type equation (12) we obtain a non-integrable difference–difference equation. In other words, a necessary condition for the integrability of equation (12) is that  $\ell = \infty$ .

### 3.2. A non-integrable difference–difference Toda equation

The multiscale expansion of the standard Toda lattice,

$$\ddot{u}_n = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}), \tag{13}$$

has been carried out by Kalyakin in [20], after the publication of the pioneering work by Zakharov and Kuznetsov [40]. Here  $u_n = u_n(t)$ , and by dot we mean the time derivative.

We shall perform the multiscale analysis of a non-integrable dispersive discretization of equation (13). It reads

$$u_{n,m+1} - 2u_{n,m} + u_{n,m-1} = a[\exp(u_{n-1,m} - u_{n,m}) - \exp(u_{n,m} - u_{n+1,m})], \tag{14}$$

where  $a \in \mathbb{R}$  is a parameter related to the discretization of the time variable.

We shall proceed, as in the previous example, by considering equation (14) for small amplitudes  $u_{n,m} = \epsilon w_{n,m}$ . In this case the linear part of the resulting equation admits the dispersion relation

$$\omega(\kappa) = 2 \arccos \left[ \alpha \sin \left( \frac{\kappa}{2} \right) \right]. \tag{15}$$

Using expansion (6) we obtain a set of determining equations similar to that we obtained for the integrable difference–difference Toda equation (4), but with different coefficients.

The resulting NLS-type equation for the harmonic  $w_0^{(1)}$ , obtained at  $\mathcal{O}(\epsilon^2)$ , is again given by equation (12) with the following real coefficients:

$$\tau_9 = \frac{v_g^2 \cos \omega - a \cos \kappa}{2 \sin \omega},$$

$$\tau_{10} = \frac{a(\cos \kappa - 1)}{\sin \omega} \left[ \frac{2a(\cos \kappa - 1)}{v_g^2 - a} + \cos \kappa - 1 + \frac{\sin^2 \kappa}{(a - 1)(\cos \kappa - 1)} \right],$$

where  $v_g = d\omega/d\kappa$  is the group velocity corresponding to the dispersion relation (15). It is remarkable to note that this NLS-type equation is integrable (if  $\ell = \infty$ ) even if the starting equation (14) is not. In the next paragraph we illustrate further this fact with two examples of non-integrable difference equations that lead to, respectively, an integrable or non-integrable NLS-type equation. We also give some clues to what the cause of this discrepancy could be.

### 3.3. Non-integrable difference–difference KdV equations

We now present the multiscale analysis of two different non-integrable lattice KdV equations obtained discretizing the continuous KdV equation. They read:

$$u_{n,m+1} - u_{n,m-1} = \frac{a}{4}(u_{n+3,m} - 3u_{n+1,m} + 3u_{n-1,m} - u_{n-3,m}) - \frac{b}{2}(u_{n+1,m}^2 - u_{n-1,m}^2), \tag{16}$$

and

$$u_{n,m+1} - u_{n,m-1} = \frac{a}{4}(u_{n+3,m} - 3u_{n+1,m} + 3u_{n-1,m} - u_{n-3,m}) - \frac{b}{2}(u_{n+1,m}^2 - u_{n,m}^2), \tag{17}$$

where  $a, b$  are real parameters. Clearly the difference between equations (16) and (17) lies just in their nonlinear part. Precisely equation (16) has a symmetric nonlinear part, while it is asymmetric in equation (17). Since the linear part of both equations (16) and (17) is the same, they have the same dispersion relation given by

$$\omega(\kappa) = \arcsin(a \sin^3 \kappa).$$

Let us start with the analysis of equation (16) by looking at small amplitudes solutions  $u_{n,m} = \epsilon w_{n,m}$  and considering expansion (6).



At  $\mathcal{O}(1)$  we obtain a set of determining equations which give  $w_0^{(\alpha)} = 0$  for  $|\alpha| > 1$ , while the  $\mathcal{O}(\epsilon)$  gives  $w_0^{(0)} = 0$  and the following determining equations:

$$(v_g \delta_{n_1} + \delta_{m_1}) w_0^{(1)} = 0, \tag{18}$$

$$w_1^{(2)} = \sigma_1 (w_0^{(1)})^2, \tag{19}$$

where

$$v_g = \frac{d\omega}{d\kappa}, \quad \sigma_1 = -\frac{b \cos \kappa}{2a \sin^2 \kappa (4 \cos^3 \kappa - \cos \omega)}.$$

As in the previous examples, equation (18) is solved by  $w_0^{(1)}(n_1, \{m_i\}_{i=1}^K) = w_0^{(1)}(n_2, \{m_i\}_{i=2}^K)$  with  $n_2 = n_1 - v_g m_1$ . Moreover, higher harmonics imply that  $w_1^{(\alpha)} = 0$  if  $|\alpha| \geq 3$ .

At  $\mathcal{O}(\epsilon^2)$  we obtain

$$w_1^{(0)} = \sigma_2 |w_0^{(1)}|^2, \quad \sigma_2 = \frac{b}{v_g}, \tag{20}$$

for  $\alpha = 0$ . For  $\alpha = 1$ , after removing secularities and using equations (19), (20) we find the following NLS-type equation:

$$i \delta_{m_2} w_0^{(1)} = \sigma_3 \delta_{n_2}^2 w_0^{(1)} + \sigma_4 w_0^{(1)} |w_0^{(1)}|^2, \tag{21}$$

where

$$\sigma_3 = \frac{a \sin \kappa [3(1 - 3 \cos^2 \kappa) - v_g^2 (1 - \cos^2 \kappa)]}{2 \cos \omega},$$

$$\sigma_4 = \frac{b^2}{a \cos \omega \sin \kappa} \left[ \frac{\cos \omega}{3 \cos \kappa} + \frac{\cos \kappa}{2(\cos \omega - 4 \cos^3 \kappa)} \right].$$

Equation (21) is again an integrable NLS equation (if  $\ell = \infty$ ) even if the starting difference–difference KdV equation (16) is non-integrable. At this level the integrability of equation (21) is definitely due to the symmetric version of the nonlinear part of equation (16). As a matter of fact, the same multiscale expansion carried out on equation (17) yields a NLS-type equation of the form (21), but with complex coefficients  $\sigma_3, \sigma_4$ , thus breaking its integrability property. In this latter case the coefficients  $\sigma_i, 1 \leq i \leq 4$ , read

$$\sigma_1 = \frac{b \exp(i\kappa)}{4a \sin^2 \kappa (4 \cos^3 \kappa - \cos \omega)}, \quad \sigma_2 = \frac{b}{2v_g},$$

and

$$\sigma_3 = \frac{a \sin \kappa [3 - 9 \cos^2 \kappa - v_g^2 \sin^2 \kappa]}{\cos \omega}, \quad \sigma_4 = \frac{ib(\sigma_1 + \sigma_2)[1 - \exp(i\kappa)]}{2 \cos \omega}.$$

#### 4. Multiscale expansion of linearizable discrete equations

In this section we shall consider difference–difference and differential–difference equations, which are linearized by a Miura transformation. In particular, we analyse a differential–difference Burgers equation and the Hietarinta equation. Let us here recall that a multiscale reduction of the Hietarinta equation has been already considered in [26] for  $\ell = 2$ , while its linearizability has been proven in [31].

#### 4.1. Discrete Burgers equations

We look for a discrete Burgers equation which has a real dispersion relation. This request is equivalent to saying that its linear part admits dispersive wave solutions. A discrete Burgers equation has been derived in [13], but it is not dispersive since it involves an asymmetric discrete-time derivative. One can prove that a dispersive completely discrete Burgers-type equation which is linearizable via a discrete Cole–Hopf map does not exist. Roughly speaking this is related to the fact that integrability implies non-symmetric discrete-time derivatives, which are usually incompatible with real dispersion relations.

Therefore we shall consider the following dispersive differential–difference Burgers equation [27]:

$$ia^2 \dot{u}_n = (1 + au_n)(u_{n+1} - u_n) + \frac{u_{n-1} - u_n}{1 + au_{n-1}}, \quad (22)$$

with  $a \in \mathbb{R}$ . Equation (22) is obtained as a compatibility condition for the following Lax pair:

$$\Psi_{n+1}(t) = (1 + au_n)\Psi_n(t), \quad i\dot{\Psi}_n(t) = \frac{u_n - u_{n-1} + au_n u_{n-1}}{a(1 + au_{n-1})}\Psi_n(t). \quad (23)$$

As equation (22) has a real dispersion relation we can use the same procedure described in section 2, but with a continuous-time variable. The linear part of equation (22) admits wave solutions with a dispersion relation given by

$$\omega(\kappa) = \frac{2}{a^2}(\cos \kappa - 1).$$

For solutions of equation (22) with small complex amplitude  $u_n(t) = \epsilon w_n(t)$ , where  $w_n(t)$  is expanded according to a natural modification of equation (6), we obtain at  $\mathcal{O}(1)$  the condition  $w_0^{(\alpha)} = 0$  for  $\alpha > 1$  and  $\alpha < 0$ .

At  $\mathcal{O}(\epsilon)$  we have  $w_1^{(\alpha)} = 0$  for  $\alpha > 2$  and  $\alpha < 0$  and  $w_0^{(0)} = 0$ . Moreover we obtain

$$(v_g \delta_{n_1} + \partial_{t_1})w_0^{(1)} = 0,$$

whose solution is  $w_0^{(1)}(n_1, \{t_i\}_{i=1}^K) = w_0^{(1)}(n_2, \{t_i\}_{i=2}^K)$  with  $n_2 = n_1 - v_g t_1$ ,  $v_g = d\omega/d\kappa$ , and

$$w_1^{(2)} = \frac{a \cos \kappa (1 - e^{-i\kappa})}{2 \cos \kappa - \cos(2\kappa) - 1} (w_0^{(1)})^2. \quad (24)$$

At  $\mathcal{O}(\epsilon^2)$  we obtain  $w_1^{(0)} = 0$  for  $\alpha = 0$ , while for  $\alpha = 1$ , using equation (24), we obtain a linear equation

$$\partial_{t_2} w_0^{(1)} = \frac{\cos \kappa}{a^2} \delta_{n_2}^2 w_0^{(1)}, \quad (25)$$

a remainder of the fact that equation (22) is linearizable.

Let us now consider the multiscale analysis of a difference–difference equation obtained through a naïve, but symmetric, discretization of equation (22). Up to our knowledge the resulting equation is not linearizable and it is not associated with a Lax pair of the type given in equation (23). It reads

$$\frac{ia^2}{2b}(u_{n,m+1} - u_{n,m-1}) = (1 + au_{n,m})(u_{n+1,m} - u_{n,m}) + \frac{u_{n-1,m} - u_{n,m}}{1 + au_{n-1,m}}, \quad (26)$$

where  $b \in \mathbb{R}$  is a parameter related to the time discretization. The linear part of equation (26) admits dispersive wave solutions with the dispersion relation

$$\omega(\kappa) = \arcsin \left[ \frac{2b}{a^2}(\cos \kappa - 1) \right].$$

By applying the usual procedure, we obtain, at  $\mathcal{O}(\epsilon^2)$ , the analogue of equation (25). It reads

$$i\delta_{m_2} w_0^{(1)} = -\frac{v_g [v_g^2 (\cos \kappa - 1) - \cos \kappa]}{2 \sin \kappa} \delta_{n_2}^2 w_0^{(1)},$$

which is again a linear equation.

#### 4.2. The Hietarinta equation

The  $\mathbb{Z}^2$ -lattice equation

$$\frac{u_{n,m} + e_2}{u_{n,m} + e_1} \frac{u_{n+1,m+1} + o_2}{u_{n+1,m+1} + o_1} = \frac{u_{n+1,m} + e_2}{u_{n+1,m} + o_1} \frac{u_{n,m+1} + o_2}{u_{n,m+1} + e_1} \quad (27)$$

has been introduced by Hietarinta in [14]. Here,  $e_i$  and  $o_i$ ,  $i = 1, 2$ , are real and distinct parameters. As proven in [31], equation (27) is linearizable. For our purposes it is convenient to reparametrize the coefficients  $e_i$  and  $o_i$  by means of the following transformations:  $e_i \mapsto e_i^{-1}$  and  $o_i \mapsto o_i^{-1}$ ,  $i = 1, 2$ .

As noted in [26], the linear part of equation (27) has travelling wave solutions  $u_{n,m} = \exp\{i[\kappa n - \omega(\kappa)m]\}$  with a complex dispersion relation. Nevertheless, the constraint  $o_1 + e_1 = o_2 + e_2$  provides the real dispersion relation

$$\omega(\kappa) = 2 \arctan \left[ \frac{a-b}{a+b} \tan \left( \frac{\kappa}{2} \right) \right],$$

where  $a = o_2 - o_1$ ,  $b = e_1 - e_2$ .

For small amplitude solutions  $u_{n,m} = \epsilon w_{n,m}$ , with  $w_{n,m}$  given by equation (6) we obtain from equation (27) a set of equations which, at  $\mathcal{O}(1)$ , give the condition  $w_0^{(\alpha)} = 0$  for  $\alpha > 1$  and  $\alpha < 0$ .

At  $\mathcal{O}(\epsilon)$  we find  $w_1^{(0)} = 0$  and

$$(v_g \delta_{n_1} + \partial_{m_1}) w_1^{(1)} = 0, \quad (28)$$

$$w_2^{(2)} = \zeta_1 (w_1^{(1)})^2, \quad (29)$$

where

$$v_g = \frac{d\omega}{d\kappa}, \quad \zeta_1 = e_1 + b \frac{\exp(i\kappa) + \exp[i(\kappa + \omega)]}{\exp(i\kappa) - \exp(i\omega)}.$$

Equation (28) is solved by  $w_1^{(1)}(n_1, \{t_i\}_{i=1}^K) = w_1^{(1)}(n_2, \{t_i\}_{i=2}^K)$  with  $n_2 = n_1 - v_g t_1$ .

At order  $\mathcal{O}(\epsilon^2)$ , by taking into account equation (29), we obtain for  $\alpha = 0$ ,

$$w_2^{(0)} = (2e_1 + b - a) |w_1^{(1)}|^2,$$

which leads, for  $\alpha = 1$ , to the following linear equation

$$i\delta_{m_2} w_1^{(1)} = \zeta_3 \delta_{n_2}^2 w_1^{(1)}, \quad \zeta_3 = \frac{1}{2} (\cos \kappa - \cos \omega) \sin \omega. \quad (30)$$

In our previous work [26] the multiscale analysis of the Hietarinta equation provided, in the  $\ell = 2$  case, the following difference–difference equation:

$$\begin{aligned} & i(\phi_{n_2, m_2+1} - \phi_{n_2, m_2}) + c_1(\phi_{n_2+2, m_2} - 2\phi_{n_2, m_2} + \phi_{n_2-2, m_2}) \\ & + c_2(\phi_{n_2+1, m_2} - 2\phi_{n_2, m_2} + \phi_{n_2-1, m_2}) + c_3 \phi_{n_2, m_2} |\phi_{n_2, m_2}|^2 \\ & + c_4 \psi_{n_2, m_2} \phi_{n_2, m_2} + c_5 (\phi_{n_2, m_2})^2 \bar{\phi}_{n_2, m_2} = 0, \end{aligned} \quad (31)$$

where  $\phi = w_1^{(1)}$  and the function  $\psi$  is defined through the equation

$$\psi_{n_2+1,m_2} - \psi_{n_2-1,m_2} = c_6 [\bar{\phi}_{n_2,m_2} (\phi_{n_2+1,m_2} - \phi_{n_2-1,m_2}) + \phi_{n_2,m_2} (\bar{\phi}_{n_2+1,m_2} - \bar{\phi}_{n_2-1,m_2})]. \quad (32)$$

The coefficients  $c_i$ ,  $1 \leq i \leq 6$ , have been determined and they are all real. Equation (31) is a (non-integrable) discrete integral equation due to the presence of the function  $\psi$ . It is remarkable to note that, if  $\ell = \infty$ , equation (32) can be explicitly integrated and the resulting coefficients  $c_i$  combine in such a way that the nonlinear term in equation (31) vanishes, thus giving equation (30).

## 5. Concluding remarks

In this paper we presented a conjecture which gives necessary conditions for the integrability and linearizability of dispersive difference–difference and differential–difference equations. This conjecture has been confirmed by a long list of examples, contained in sections 3, 4. In particular, performing the discrete multiscale analysis, we found that the resulting NLS-type equation turns out to be integrable if

- (1) the starting  $\mathbb{Z}^2$ -lattice equation is integrable, or
- (2) the starting  $\mathbb{Z}^2$ -lattice equation is non-integrable, but obtained through a symmetric discretization of a continuous integrable equation.

We expect that claim (2) fails at higher orders of the multiscale expansion. Moreover the resulting NLS-type equation turns out to be non-integrable if the starting nonlinear  $\mathbb{Z}^2$ -lattice equation is obtained through a non-symmetric discretization of a continuous integrable equation. Let us recall that discrete symmetric derivatives play a fundamental role in the integrability of discrete equations, see [39]. We stress here that the integrability of the NLS-type equation is lost as soon as we go to a finite slow varyness order since shift operators do not satisfy the Leibniz rule and thus the reduced Lax pair is no longer compatible.

Concerning the linearizability of  $\mathbb{Z}^2$ -lattice equations we obtained a linear Schrödinger equation if:

- (1) the starting  $\mathbb{Z}^2$ -lattice equation is linearizable, or
- (2) the starting  $\mathbb{Z}^2$ -lattice equation is obtained through a discretization of a continuous linearizable equation.

The above characterization is evident just by considering expansions on functions of infinite order of slow varyness. As soon as we choose a finite order, as shown for  $\ell = 2$ , we may obtain nonlocal discrete equations as the integration of the equations defining the auxiliary harmonics may not be exact [26].

This work still leaves many open problems. Work is in progress on

- The proof of the conjecture for some relevant classes of lattice equations, as for example the class of dispersive affine linear  $\mathbb{Z}^2$ -lattice equation on the square.
- The construction of higher orders in the multiscale expansion and on the definition of an order of integrability of a lattice equation (in the spirit of the paper [8]).

It is worthwhile noticing that a meaningful classification of integrable  $\mathbb{Z}^2$ -lattice equations can be obtained only by going at least to the fourth order in the multiscale expansion (indeed, the third order is not enough to show the non-integrability of the discretized KdV equation (16)).

In the present paper we have considered the modulation of weak plane waves solutions of the discrete systems. Work is also in progress on the modulation of constant solutions. In such a case the multiscale analysis should provide a KdV-type equation instead of a NLS-type one [24].

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## References

- [1] Abarenkova N, Anglès d'Auriac J C, Boukraa S, Hassani S and Maillard J M 2000 *J. Phys. A: Math. Gen.* **33** 1465–78
- [2] Ablowitz M J, Halburd R and Herbst B 2000 *Nonlin.* **13** 889–905
- [3] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Providence, RI: SIAM)
- [4] Agarwal R P 1992 *Difference Equations and Inequalities: Theory, Methods and Applications* (New York: Dekker)
- [5] Agrotis M, Lafortune S and Kevrekidis P G *Discr. Cont. Dyn. Syst.* **2005** (supp.) 22–9
- [6] Calogero F and Eckhaus W 1987 *Inverse Problems* **3** 229–62  
Calogero F and Eckhaus W 1987 *Inverse Problems* **4** 11–33
- [7] Campa A, Giansanti A, Tenenbaum A, Levi D and Ragnisco O 1993 *Phys. Rev. B* **48** 10168–76
- [8] Degasperis A and Procesi M 1999 *Asymptotic Integrability in Symmetry and Perturbation Theory SPT98* (Singapore: World Scientific) pp 23–37
- [9] Elaydi S N 1999 *An Introduction to Difference Equations* (New York: Springer)
- [10] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825–32
- [11] Hernandez Heredero R, Levi D, Petrera M and Scimiterna C 2007 *J. Phys. A: Math. Theor.* **40** F831–40
- [12] Hernandez Heredero R, Levi D, Petrera M and Scimiterna C *J. Nonlin. Math. Phys.* submitted, <http://www.arxiv.org/abs/0709.3704>
- [13] Hernandez Heredero R, Levi D and Winternitz P 1999 *J. Phys. A: Math. Gen.* **32** 2685–95
- [14] Hietarinta J 2004 *J. Phys. A: Math. Gen.* **37** L67–73
- [15] Hietarinta J and Viallet C 1998 *Phys. Rev. Lett.* **81** 326–8
- [16] Hietarinta J and Viallet C 2007 *J. Phys. A: Math. Theor.* **40** 12629–43
- [17] Hirota R 1977 *J. Phys. Soc. Japan* **43** 2074–8
- [18] Ji X-D, Levi D and Petrera M 2007 Multiscale expansion of the lattice sine-Gordon equation *Difference Equations, Special Functions and Orthogonal Polynomials* ed S Elaydi (Singapore: World Scientific) pp 305–14
- [19] Jordan C *et al* 1939 *Calculus of Finite Differences* (Sopron: Röttig and Romwalter)
- [20] Kalyakin L A 1988 *Theor. Math. Phys.* **76** 891–4
- [21] Kelley W G and Peterson A C 1991 *Difference Equations: An Introduction With Applications* (New York: Academic)
- [22] Leon J and Manna M 1999 *J. Phys. A: Math. Gen.* **32** 2845–69
- [23] Levi D 2005 *J. Phys. A: Math. Gen.* **38** 7677–85
- [24] Levi D and Hernández Heredero R 2005 *J. Nonlin. Math. Phys.* **12** 440–55
- [25] Levi D and Martina L 2001 *J. Phys. A: Math. Gen.* **34** 10357–68
- [26] Levi D and Petrera M 2006 *J. Math. Phys.* **47** 043509
- [27] Levi D, Ragnisco O and Bruschi M 1983 *Nuovo Cimento B* **11** 33–51
- [28] MacKay R S and Aubry S 1994 *Nonlinearity* **7** 1623–43
- [29] Mikhailov A V, Shabat A B and Sokolov V V 1991 *What is Integrability?* ed V E Zakharov (Berlin: Springer)
- [30] Papageorgiou V G 2006 private communication
- [31] Ramani A, Joshi N, Grammaticos B and Tamizhmani T 2006 *J. Phys. A: Math. Gen.* **39** L145–9
- [32] Roberts J A G and Vivaldi F 2003 *Phys. Rev. Lett.* **90** 034102
- [33] Taniuti T 1974 *Prog. Theor. Phys.* **55** 1654–76
- [34] Taniuti T and Nishihara K 1983 *Nonlinear Waves* (Boston: Pitman)
- [35] Toda M 1988 *Theory of Nonlinear Lattice* (New York: Springer)
- [36] Veselov A P 1991 *Russ. Math. Surveys* **46** 1–51
- [37] Veselov A P 1992 *Commun. Math. Phys.* **145** 181–93
- [38] Viallet C <http://arxiv.org/abs/math-ph/0609043>
- [39] Yamilov R 2006 *J. Phys. A: Math. Gen.* **39** R541–623
- [40] Zakharov V E and Kuznetsov E A 1986 *Phys. D* **18** 455–63